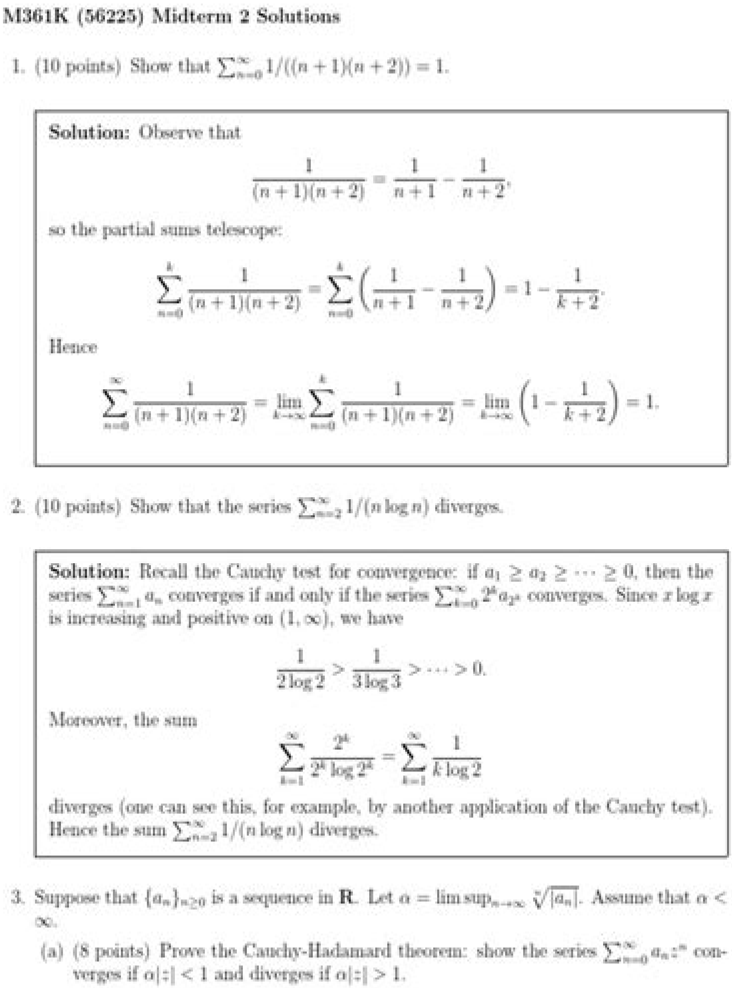
Real Analysis Problems And Solutions



M361K (56225) Midterm 2 Solutions

1. (10 points) Show that $\sum_{n=0}^{\infty} 1/((n+1)(n+2)) = 1$.

Solution: Observe that

$$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2},$$

so the partial sums telescope:

$$\sum_{n=0}^{k} \frac{1}{(n+1)(n+2)} = \sum_{n=0}^{k} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) = 1 - \frac{1}{k+2}.$$

Hence

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = \lim_{k\to\infty} \sum_{n=0}^{k} \frac{1}{(n+1)(n+2)} = \lim_{k\to\infty} \left(1 - \frac{1}{k+2}\right) = 1.$$

2. (10 points) Show that the series $\sum_{n=2}^{\infty} 1/(n \log n)$ diverges.

Solution: Recall the Cauchy test for convergence: if $a_1 \geq a_2 \geq \cdots \geq 0$, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges. Since $x \log x$ is increasing and positive on $(1, \infty)$, we have

$$\frac{1}{2\log 2} > \frac{1}{3 \log 3} > \cdots > 0.$$

Moreover, the sum

$$\sum_{k=1}^{\infty} \frac{2^k}{2^k \log 2^k} = \sum_{k=1}^{\infty} \frac{1}{k \log 2}$$

diverges (one can see this, for example, by another application of the Cauchy test). Hence the sum $\sum_{n=2}^{\infty} 1/(n \log n)$ diverges.

3. Suppose that $\{a_n\}_{n\geq 0}$ is a sequence in $\mathbf{R}$. Let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Assume that $\alpha < \infty$.

   (a) (8 points) Prove the Cauchy-Hadamard theorem: show the series $\sum_{n=0}^{\infty} a_n z^n$ converges if $\alpha|z| < 1$ and diverges if $\alpha|z| > 1$.

Real analysis problems and solutions form the backbone of mathematical rigor and logical reasoning in the field of mathematics. Real analysis is focused on the properties of real numbers, sequences and series, functions, limits, continuity, differentiation, integration, and more. This article explores various problems encountered in real analysis and provides solutions to enhance understanding and mastery of the subject. By dissecting these problems, we can appreciate the beauty and complexity of real analysis.

Fundamental Concepts in Real Analysis

Before delving into specific problems, it’s essential to grasp some fundamental concepts in real analysis:

1. Sequences and Convergence

A sequence is a list of numbers arranged in a specific order. The convergence of a sequence refers to the behavior of the sequence as it progresses to infinity. A sequence (a_n) converges to a limit L if, for every $\epsilon > 0$, there exists an integer N such that for all $n > N$, $|a_n - L| < \epsilon$.

2. Series and Summation

A series is the sum of the terms of a sequence. A series converges if the sequence of its partial sums converges to a limit. The convergence of series is often analyzed using tests like the Ratio Test, Root Test, and Comparison Test.

3. Continuity and Limits of Functions

A function $f(x)$ is continuous at a point c if:
- $f(c)$ is defined.
- $\lim_{x \to c} f(x)$ exists.
- $\lim_{x \to c} f(x) = f(c)$.

Common Problems and Solutions

Problem 1: Determine the Convergence of a Sequence

Consider the sequence defined by $a_n = \frac{1}{n}$. Does this sequence converge? If so, what is its limit?

Solution:
To determine the convergence, we examine the limit as n approaches infinity:
$$
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0.
$$
For every $\epsilon > 0$, we can choose N such that for all $n > N$, $|a_n - 0| < \epsilon$. Specifically, choose $N = \frac{1}{\epsilon}$. Hence, the sequence converges to 0.

Problem 2: Series Convergence Test

Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

converges.

Solution:
We can apply the p-series test. A p-series converges if $p > 1$. Here, we have $p = 2$:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges since } p = 2 > 1.$$

Thus, the series converges.

Problem 3: Continuity of a Function

Show that the function $f(x) = \sqrt{x}$ is continuous at $x = 4$.

Solution:
We need to verify the three conditions for continuity at $x = 4$:
1. $f(4)$ is defined: $f(4) = \sqrt{4} = 2$.
2. The limit exists:

$$\lim_{x \to 4} f(x) = \lim_{x \to 4} \sqrt{x} = \sqrt{4} = 2.$$

3. The limit equals the function value:

$$\lim_{x \to 4} f(x) = f(4).$$

Since all conditions are satisfied, f is continuous at $x = 4$.

Problem 4: The Bolzano-Weierstrass Theorem

Prove that every bounded sequence has a convergent subsequence.

Solution:
Let (a_n) be a bounded sequence. By the Bolzano-Weierstrass theorem, since the sequence is bounded, it must have at least one limit point in $\mathbb{R}$. Thus, we can extract a subsequence (a_{n_k}) that converges to this limit point. The construction of this subsequence involves choosing terms of the original sequence that approach the limit point.

Problem 5: Uniform Convergence

Determine whether the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges uniformly on $[0, 1)$.

Solution:
To check for uniform convergence, we can use the Weierstrass M-test. We note that $|f_n(x)| = \frac{|x|^n}{n} \leq \frac{1}{n}$ for $x \in [0, 1)$. The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but the series converges for fixed $x < 1$. To show uniform convergence, we must analyze

the supremum of $|f_n(x)|$ over $[0, 1)$. Therefore, the series converges uniformly on compact subsets of $[0, 1)$.

Advanced Problems in Real Analysis

Problem 6: Finding Limits of Functions

Evaluate $\lim_{x \to 0} \frac{\sin(x)}{x}$.

Solution:
Using L'Hôpital's Rule (since both the numerator and denominator approach 0 as $x \to 0$):
$$
\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = \cos(0) = 1.
$$

Problem 7: Riemann Integral

Show that the function $f(x) = x^2$ is Riemann integrable on the interval $[0, 1]$.

Solution:
A function is Riemann integrable if it is bounded and its set of discontinuities has measure zero. The function $f(x) = x^2$ is continuous on $[0, 1]$, hence it is bounded and has no discontinuities. Therefore, f is Riemann integrable.

Problem 8: Mean Value Theorem

Use the Mean Value Theorem to show that $f(x) = x^2$ has a point $c \in (0, 1)$ such that $f'(c) = 1$.

Solution:
The Mean Value Theorem states that for a continuous function f on $[a, b]$ and differentiable on (a, b), there exists a $c \in (a, b)$ such that:
$$
f'(c) = \frac{f(b) - f(a)}{b - a}.
$$
Let $a = 0$ and $b = 1$:
$$
f(1) - f(0) = 1^2 - 0^2 = 1.
$$
Thus,
$$

f'(c) = \frac{1 - 0}{1 - 0} = 1.
\]
Calculating \(f'(x) = 2x \), we set \(2c = 1 \) leading to \(c = \frac{1}{2} \in (0, 1) \).

Conclusion

Real analysis problems and solutions not only enhance mathematical understanding but also build critical thinking skills necessary for advanced mathematics. The problems explored herein range from basic convergence concepts to more advanced topics like uniform convergence and the Riemann integral. Mastery of these topics provides a solid foundation for further study in analysis, topology, and applied mathematics. By systematically working through these problems, one can develop a deeper appreciation for the intricacies of real analysis.

Frequently Asked Questions

What is the significance of the Bolzano-Weierstrass theorem in real analysis?

The Bolzano-Weierstrass theorem states that every bounded sequence in $\mathbb{R}$ has a convergent subsequence. This is significant as it establishes compactness in real analysis and is fundamental in proving various results related to continuity and limits.

How do you prove that a function is uniformly continuous?

To prove that a function f is uniformly continuous on an interval [a, b], you need to show that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x, y in [a, b], if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$, regardless of the choice of x and y.

What is the difference between pointwise and uniform convergence of sequences of functions?

Pointwise convergence occurs when, for each point in the domain, the sequence of function values converges to a limit. Uniform convergence means that the speed of convergence is uniform across the entire domain, allowing interchange of limits and integrals under certain conditions.

Can you explain what a Cauchy sequence is and its

importance?

A Cauchy sequence is a sequence where for every ε > 0, there exists an N such that for all m, n > N, the distance between the terms $|a_m - a_n| < \varepsilon$. It is important because it characterizes convergence in complete spaces; every Cauchy sequence converges in complete metric spaces like ℝ.

What are the criteria for a function to be Riemann integrable?

A function is Riemann integrable on a closed interval [a, b] if it is bounded and its set of discontinuities has Lebesgue measure zero. This means that the 'bad' points where the function is discontinuous do not take up 'space' in the interval.

How do you approach solving real analysis problems involving limits?

To solve limits in real analysis, you can use techniques such as direct substitution, factoring, rationalization, L'Hôpital's Rule for indeterminate forms, and analyzing the behavior of functions as they approach the limit.

What is the purpose of the ε-δ definition of limits in real analysis?

The ε-δ definition of limits provides a rigorous way to define the concept of a limit. It states that the limit of f(x) as x approaches c is L if for every ε > 0, there exists a δ > 0 such that whenever $|x - c| < \delta$, it follows that $|f(x) - L| < \varepsilon$. This precise formulation is fundamental for proving theorems in analysis.

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