

Prove By Mathematical Induction Problems

Proving a Summation Formula by Mathematical Induction

Example: Show that: $\sum_{i=1}^n = \frac{n(n+1)}{2}$

Solution:

Note: Once we have this conjecture, mathematical induction can be used to prove it correct.

- **BASIS STEP:** $P(1)$ is true since $1(1+1)/2 = 1$.
- **INDUCTIVE STEP:** Assume true for $P(k)$.

The inductive hypothesis is $\sum_{i=1}^k = \frac{k(k+1)}{2}$

Under this assumption,

$$\begin{aligned} 1 + 2 + \dots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

Proving by mathematical induction is a fundamental technique in mathematics that allows us to establish the validity of infinite sequences of statements. This method is particularly useful for proving assertions about natural numbers, sequences, and inequalities, among other topics. In this article, we will explore the principles of mathematical induction, the steps involved in conducting a proof, common examples, and some problems for practice.

Understanding Mathematical Induction

Mathematical induction is a method of proof that is widely used in mathematics. It is based on the principle that if a statement holds for a certain base case and can be shown to hold for an arbitrary case assuming it holds for the previous case, it must hold for all natural numbers.

The Principle of Mathematical Induction

Mathematical induction consists of two main steps:

1. **Base Case:** Verify that the statement is true for the initial value, usually $(n = 1)$.
2. **Inductive Step:** Assume that the statement is true for some arbitrary natural number (k) (this is called the inductive hypothesis), and then prove that the statement is also true for $(k + 1)$.

If both steps are successfully completed, we can conclude that the statement is true for all natural numbers $\forall (n \in \mathbb{N})$.

The Steps of a Proof by Induction

To conduct a proof by induction, follow these structured steps:

1. Identify the Statement: Clearly define the statement $\forall (P(n))$ you want to prove.
2. Base Case: Prove that $\forall (P(1))$ (or any other starting point) is true.
3. Inductive Hypothesis: Assume that $\forall (P(k))$ is true for an arbitrary natural number $\forall (k \in \mathbb{N})$.
4. Inductive Step: Prove that $\forall (P(k + 1))$ is true using the assumption that $\forall (P(k))$ is true.
5. Conclusion: State that since both the base case and inductive step have been proven, $\forall (P(n))$ is true for all natural numbers $\forall (n \in \mathbb{N})$.

Common Examples of Proof by Induction

To illustrate the process of proving by induction, let's consider a couple of classic examples.

Example 1: Sum of the First $\forall (n \in \mathbb{N})$ Natural Numbers

Statement: Prove that for all natural numbers $\forall (n \in \mathbb{N})$,

$$\forall [P(n): 1 + 2 + 3 + \dots + n = \frac{n(n + 1)}{2}]$$

Base Case: For $\forall (n = 1)$,

$$\forall [1 = \frac{1(1 + 1)}{2} = \frac{2}{2} = 1]$$

Thus, $\forall (P(1))$ is true.

Inductive Hypothesis: Assume $\forall (P(k))$ is true for some $\forall (k \in \mathbb{N})$, i.e.,

$$\forall [1 + 2 + 3 + \dots + k = \frac{k(k + 1)}{2}]$$

Inductive Step: We need to show that $\forall (P(k + 1))$ is true, i.e.,

$$\forall [1 + 2 + 3 + \dots + k + (k + 1) = \frac{(k + 1)(k + 2)}{2}]$$

Starting from the left side:

$$\begin{aligned} & 1 + 2 + 3 + \dots + k + (k + 1) = \frac{k(k + 1)}{2} + (k + 1) \\ & \end{aligned}$$

Factoring $\forall ((k + 1))$ out:

$$\begin{aligned}
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2} \\
&\]
\end{aligned}$$

Thus, $(P(k+1))$ holds.

Since both the base case and the inductive step are proven, by induction, $(P(n))$ is true for all natural numbers (n) .

Example 2: Inequality $(2^n > n^2)$

Statement: Prove that for all integers $(n \geq 5)$,

$$(P(n): 2^n > n^2)$$

Base Case: For $(n = 5)$,

$$(2^5 = 32 > 25 = 5^2)$$

Thus, $(P(5))$ is true.

Inductive Hypothesis: Assume $(P(k))$ is true for some $(k \geq 5)$, i.e., $(2^k > k^2)$.

Inductive Step: We need to show $(P(k+1))$ is true:

$$(2^{k+1} > (k+1)^2)$$

Starting from the left side,

$$\begin{aligned}
&(2^{k+1} = 2 \cdot 2^k) \\
&\]
\end{aligned}$$

Using the inductive hypothesis:

$$\begin{aligned}
&(2 \cdot k^2) \\
&\]
\end{aligned}$$

Now, we need to show that $(2 \cdot k^2 > (k+1)^2)$ for $(k \geq 5)$:

$$\begin{aligned}
&(2k^2 > (k^2 + 2k + 1)) \\
&\]
\end{aligned}$$

Rearranging gives:

$$\begin{aligned}
&(2k^2 - k^2 - 2k - 1 > 0 \implies k^2 - 2k - 1 > 0) \\
&\]
\end{aligned}$$

The roots of the equation $(k^2 - 2k - 1 = 0)$ are:

$$\begin{aligned}
&(k = 1 \pm \sqrt{2})
\end{aligned}$$

]

Since $(k \geq 5)$, this inequality is satisfied, hence $(P(k + 1))$ holds.

By induction, $(P(n))$ is true for all integers $(n \geq 5)$.

Practice Problems

To further solidify your understanding of the principle of mathematical induction, here are some problems to solve:

1. Prove that for all natural numbers (n) ,

$$[P(n): 3^n > n^3]$$

for $(n \geq 1)$.

2. Prove that for all natural numbers (n) ,

$$[P(n): 1 + 3 + 5 + \dots + (2n - 1) = n^2]$$

3. Show that for all integers $(n \geq 1)$,

$$[P(n): 5^n > n^3]$$

4. Prove that for all integers $(n \geq 0)$,

$$[P(n): 2^n \geq n + 1]$$

5. Prove that for all integers $(n \geq 1)$,

$$[P(n): n! > 2^n]$$

Conclusion

Proving by mathematical induction is a powerful tool in mathematics, allowing us to establish the truth of statements for all natural numbers. By understanding the structure of induction and practicing various problems, one can develop a strong proficiency in this proof technique. Whether you are tackling algebraic identities, inequalities, or more complex mathematical concepts, induction provides a systematic way to validate your assertions. Embrace this method, and it will serve you well in your mathematical journey.

Frequently Asked Questions

What is mathematical induction?

Mathematical induction is a proof technique used to establish that a statement is true for all natural numbers. It involves two main steps: the base case and the inductive step.

What are the steps involved in proving by mathematical induction?

The steps are: 1) Prove the base case (usually for $n=1$), 2) Assume the statement is true for $n=k$ (inductive hypothesis), and 3) Prove that the statement is true for $n=k+1$.

When is mathematical induction applicable?

It is applicable for statements that are defined for all natural numbers or a specific sequence of natural numbers, particularly those that can be expressed recursively.

Can mathematical induction be used for non-integer values?

No, mathematical induction is typically used for statements involving natural numbers or integers. It is not suitable for non-integer values.

What is a common mistake when using mathematical induction?

A common mistake is failing to correctly prove the inductive step, particularly assuming the statement holds for $n=k$ without properly showing it leads to $n=k+1$.

How do you choose the base case in mathematical induction?

The base case is often chosen as the smallest value for which the statement is claimed to be true, typically $n=1$ or $n=0$, depending on the context.

What is strong induction?

Strong induction is a variation of mathematical induction where the inductive step assumes the statement is true for all values up to k , rather than just k itself.

Can you give an example of a statement proven by induction?

An example is proving that the sum of the first n natural numbers is $n(n+1)/2$. The base case is $n=1$, and the inductive step involves showing the formula holds for $n=k+1$.

What is the significance of the inductive hypothesis?

The inductive hypothesis is crucial because it provides the assumption that the statement holds for $n=k$, which is used to demonstrate it also holds for $n=k+1$.

