Munkres Topology Solutions Chapter 3

Munkres - Topology - Chapter 3 Solutions

Section 24

Problem 24.3.

Solution: Define $g : X \to \mathbb{R}$ where $g(x) = f(x) - i_{\mathbb{R}}(x) = f(x) - x$ where $i_{\mathbb{R}}$ is the identity function. Since f and $i_{\mathbb{R}}$ are continuous, g is continuous by Theorems 18.2(e) and 21.5. Since X is connected for all three possibilities given in this problem and $\mathbb{R}$ is ordered, the intermediate-value theorem applies.

For $X = [0,1]$, observe that $g(0) \geq 0 - 0 = 0$ and $g(1) \leq 1 - 1 = 0$. Clearly if either $g(0)$ or $g(1)$ equals 0, then there is an $x_0 \in X$ where $f(x_0) - x_0 = 0$, so $f(x_0) = x_0$. Otherwise, $g(0) > 0$ and $g(1) < 0$, so by the intermediate-value theorem there is some $x_1 \in X$ where $g(x_1) = 0$, so $f(x_1) = x_1$.

The proposition is not necessarily true if $X = [0,1)$ or $X = (0,1)$. Let $f(x) = (1+x)/2$, which is obviously continuous. It follows that $f(x) = x$ if and only if $x = 1$, which not in X.

Problem 24.4.

Solution: If X has only one element, it is trivially a linear continuum, so we will assume X has at least two elements. Let $x, y \in Y$ where $x < y$. Since X is connected, $(-\infty, y)$ and (x, ∞) cannot be a separation of the space. Since the two open sets are clearly non-empty, it must be that they are not disjoint. Therefore there is some $z \in (-\infty, y) \cap (x, \infty)$, from which it follows that $z < y$ and $x < z$. We infer that $x < z < y$.

Next, let Y be a non-empty subset of X that is bounded above, and suppose Y has no supremum. Define $M = \{m \in X : m \geq y \text{ for all } y \in Y\}$ (in other words, the set of upper bounds of Y). Since Y is bounded above, M is non-empty. Then let $A = \bigcup_{y \in Y}(-\infty, y)$ and $B = \bigcup_{m \in M}(m, \infty)$. It follows that A and B are a separation of X, contradicting that X is connected. Given $x_0 \in X$, if $x_0 < y$ for some $y \in Y$, then $x_0 \in (-\infty, y) \subset A$. On the other hand, if $x_0 \geq y$ for all Y, then there is some $m \in M$ such that $m < x_0$, so $x_0 \in (m, \infty) \subset B$. Hence A and B partition X. Further, if $x_1 \in A \cap B$, then $x_1 < y$ for some $y \in Y$ and $x_1 > y'$ for all $y' \in Y$, which is impossible. Therefore A and B are disjoint. Since A and B form an impossible separation of X, we conclude that Y must have a supremum. Accordingly, X is a linear continuum.

Problem 24.4.

Solution: Given $x, y \in X \times [0,1)$ where $x < y$, we have $x = x_0 \times x_1$ and $y = y_0 \times y_1$. Since $[0,1)$ is a linear continuum, if $x_0 < y_0$, let $z_1 \in (x_1, 1)$; if $x_0 = y_0$, let $z_1 \in (x_1, y_1)$. Hence if $z = x_0 \times z_1$, then $x < z < y$.

Now let U be a non-empty subset of $X \times [0,1)$ that is bounded above. Define $M = \{m \in X \times [0,1) : m \geq a \text{ for all } a \in A\}$, which is the set of all upper bounds of A. Since A is bounded above, we're assured that M is non-empty. Designate $U = \{\pi_1(m) : m \in M\}$, which must be a non-empty subset of X. Since X is well-ordered, there is a least element x' of U. If $x' \notin \pi_1(A)$, then $x' \times 0 > a$ for all $a \in A$. Further, if $x'' < x'$, then any element of $x'' \times [0,1)$ must be less than some element of A because no upper bound of A has a first coordinate less than x'. Hence $x' \times 0$ is the supremum of A. On the other hand, if $x' \in \pi_1(A)$, the set $V = \{\pi_2(a) : a \in A\}$ is a non-empty subset of $[0,1)$ that is bounded above by 1, so it has a supremum y'. Accordingly, if $c \in [0, y')$, there is some $d > c$ such that $x' \times d \in A$, which is greater than $x' \times c$. Therefore $x' \times y'$ is the supremum of A, and $X \times [0,1)$ has the supremum property. We conclude that $X \times [0,1)$ is a linear continuum.

Problem 24.9.

Solution: Designate $X = \mathbb{R}^2 \setminus A$, and let $x, y \in X$ be given. If there is no element of A on the straight-line path in $\mathbb{R}^2$ from x to y, then there is obviously a path between the two points by exercise 24.8(a). In the non-trivial case where there is an element of A on the straight-line path between x and y, designate $D_0 = \{\theta \in [0, 2\pi) : \tan^{-1}[(\pi_2(a) - \pi_2(x))/(\pi_1(a) - \pi_1(x))] \neq \theta \text{ for all } a \in A\}$, which are all the angles around x for which there is no element of A on the straight line passing through x at that angle. We will show that D_0 is not empty. Assume the contrary is true and D_0 is empty. It would follow that for every element of the interval $[0, 2\pi)$, there is some element of A on the line at that angle. Consequently, there is an injection from $[0, 2\pi) \to A$. Because A is countable, there would be an injection from $[0, 2\pi) \to \mathbb{N}$, establishing that $[0, 2\pi)$ is countable, a contradiction. Therefore D_0 is not empty. Choose an arbitrary θ_0 from D_0.

Munkres topology solutions chapter 3 delves into the intricate concepts of continuity, compactness, and connectedness within the framework of topology. This chapter is pivotal for understanding how topological spaces behave under various mappings and how these spaces can be manipulated. In this article, we will explore the key concepts presented in Chapter 3 of Munkres' Topology, providing solutions and deeper insights that can enhance your comprehension of these fundamental topics.

Understanding Continuity in Topology

In the context of topology, continuity is a central theme. Munkres defines continuity in terms of topological spaces, which is a generalization of the notion of continuity in real analysis.

Definition of Continuity

A function $f: X \to Y$ between two topological spaces (X, τ_X) and (Y, τ_Y) is said to be continuous if for every open set $V \in \tau_Y$, the preimage $f^{-1}(V)$ is an open set in τ_X. This definition can also be expressed through the concept of neighborhoods, where f is continuous if the image of every neighborhood of a point in X under f is a neighborhood of the image of that point in Y.

Key Properties of Continuous Functions

Munkres discusses several important properties of continuous functions, including:

- **Composition of Continuous Functions:** If $f: X \to Y$ and $g: Y \to Z$ are continuous, then the composition $g \circ f: X \to Z$ is also continuous.
- **Inverse Images of Open Sets:** The inverse image of an open set under a continuous function is open.
- **Continuous Images of Compact Spaces:** If $K \subseteq X$ is compact and $f: X \to Y$ is continuous, then $f(K)$ is compact in Y.

Compactness in Topological Spaces

Compactness is another crucial concept in topology that Munkres addresses in this chapter. A topological space is compact if every open cover has a finite subcover.

Importance of Compactness

Compactness has several important implications in topology, including:

- Every continuous image of a compact space is compact.

- Every closed subset of a compact space is compact.
- Compact spaces are homeomorphic to closed and bounded subsets of Euclidean space, a result known as the Heine-Borel theorem.

Examples of Compact Spaces

Munkres provides various examples of compact spaces to illustrate the concept:

1. The closed interval \([a, b]\) in \(\mathbb{R}\) is compact.
2. The finite set is always compact regardless of the topology.
3. The product of a finite number of compact spaces is compact, as stated by Tychonoff's theorem.

Connectedness in Topological Spaces

Connectedness is the property of a space that cannot be divided into two disjoint non-empty open sets. Munkres emphasizes the importance of connectedness in understanding the structure of topological spaces.

Definition of Connectedness

A topological space \(X \) is connected if it cannot be expressed as the union of two non-empty disjoint open sets. If such a separation exists, the space is said to be disconnected.

Types of Connected Spaces

Munkres categorizes connected spaces into several types:

- **Path-Connected:** A space is path-connected if any two points can be connected by a continuous path.
- **Locally Connected:** A space is locally connected if every point has a neighborhood base consisting of connected sets.

- **Simply Connected:** A simply connected space is path-connected and every loop can be continuously contracted to a point.

Techniques for Proving Compactness and Connectedness

In Chapter 3, Munkres provides various techniques and theorems for proving the compactness and connectedness of spaces.

Proving Compactness

To show that a space is compact, one may employ the following strategies:

1. Demonstrate that every open cover has a finite subcover directly.
2. Use the property that continuous images of compact spaces are compact.
3. Utilize the fact that closed subsets of compact spaces are compact.

Proving Connectedness

To establish that a space is connected, consider these methods:

- Assume the space is disconnected and derive a contradiction.
- Show that the image of a connected space under a continuous function is connected.
- Prove that a space is connected by using the intersection of connected subsets.

Applications of Compactness and Connectedness

The concepts of compactness and connectedness have far-reaching implications in various fields of mathematics, including analysis, algebra, and geometry.

Applications in Analysis

In real analysis, compactness is used in the proof of the extreme value theorem, which states that a continuous function on a compact interval attains its maximum and minimum values. Connectedness is crucial in understanding the properties of continuous functions and the behavior of real-valued functions over their domains.

Applications in Algebraic Topology

In algebraic topology, the study of connectedness helps in classifying spaces and understanding their topological properties. Compact spaces often arise in the study of manifolds and their properties, providing a foundation for more advanced topics in topology.

Conclusion

Munkres topology solutions chapter 3 provides a comprehensive examination of continuity, compactness, and connectedness in topological spaces. By understanding these concepts and their implications, students and mathematicians can develop a deeper appreciation for the beauty of topology and its applications across various fields. The rigorous approach Munkres takes not only clarifies fundamental principles but also equips readers with the tools needed to tackle more complex topics in topology and beyond.

Frequently Asked Questions

What are the key concepts introduced in Chapter 3 of Munkres' Topology?

Chapter 3 focuses on the concepts of basis for a topology, subbasis, and the generation of topological spaces from these elements.

How do you define a basis for a topology in Munkres' Chapter 3?

A basis for a topology on a set X is a collection of open sets such that every open set in the topology can be expressed as a union of some of the basis elements.

What is the relationship between a basis and a

subbasis in topology?

A subbasis is a collection of sets whose finite intersections form a basis for a topology, meaning the open sets of the topology can be generated from the subbasis.

Can you provide an example of how to construct a topology from a basis?

For example, if you take X = {1, 2, 3} and B = {{1}, {2, 3}}, the topology generated by B will include the empty set, the sets {1}, {2, 3}, and the entire set {1, 2, 3} as open sets.

What is the significance of the closure of a set in the context of bases?

The closure of a set is important because it helps to understand how limits and accumulation points behave within the topology defined by a basis.

How does Munkres define open sets in terms of bases?

Open sets are defined as any union of basis elements, which means an open set can be formed by taking any collection of basis sets, irrespective of whether they are finite or infinite.

What is a common mistake when dealing with bases and subbases?

A common mistake is assuming that a collection of sets is a basis simply because it covers the space; it must also satisfy the condition that intersections of basis elements yield other basis elements.

How do exercises in Chapter 3 help in understanding topological concepts?

The exercises encourage the application of definitions and theorems from the chapter, reinforcing understanding through practical problems that require constructing and manipulating topologies.

What is the role of the 'standard topology' discussed in Munkres' Chapter 3?

The standard topology on Euclidean spaces is an example of a topology generated by open balls, which serves as a foundational example for understanding bases and topological structures.

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