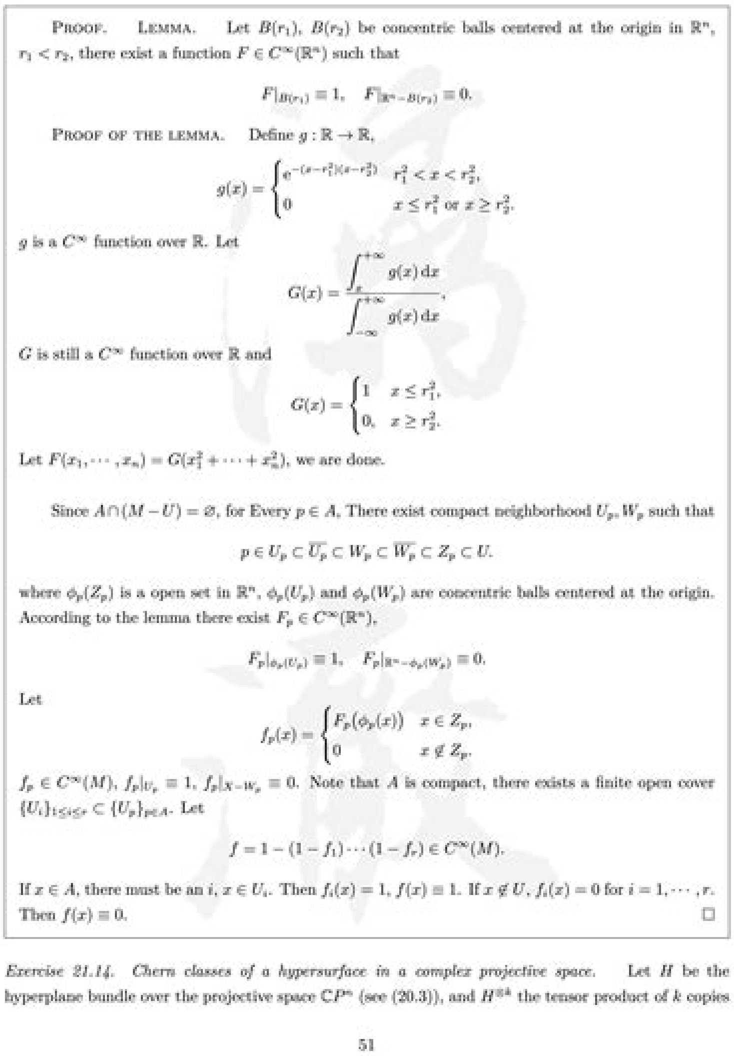
Differential Forms In Algebraic Topology



PROOF. LEMMA. Let $B(r_1)$, $B(r_2)$ be concentric balls centered at the origin in $\mathbb{R}^n$, $r_1 < r_2$, there exist a function $F \in C^\infty(\mathbb{R}^n)$ such that

$$F|_{B(r_1)} \equiv 1, \quad F|_{\mathbb{R}^n - B(r_2)} \equiv 0.$$

PROOF OF THE LEMMA. Define $g : \mathbb{R} \to \mathbb{R}$,

$$g(x) = \begin{cases} e^{-(x-r_1^2)(x-r_2^2)} & r_1^2 < x < r_2^2, \\ 0 & x \le r_1^2 \text{ or } x \ge r_2^2. \end{cases}$$

g is a C^∞ function over $\mathbb{R}$. Let

$$G(x) = \frac{\int_x^{+\infty} g(x)\,\mathrm{d}x}{\int_{-\infty}^{+\infty} g(x)\,\mathrm{d}x},$$

G is still a C^∞ function over $\mathbb{R}$ and

$$G(x) = \begin{cases} 1 & x \le r_1^2, \\ 0, & x \ge r_2^2. \end{cases}$$

Let $F(x_1, \cdots, x_n) = G(x_1^2 + \cdots + x_n^2)$, we are done.

Since $A \cap (M - U) = \varnothing$, for Every $p \in A$, There exist compact neighborhood U_p, W_p such that

$$p \in U_p \subset \overline{U_p} \subset W_p \subset \overline{W_p} \subset Z_p \subset U.$$

where $\phi_p(Z_p)$ is a open set in $\mathbb{R}^n$, $\phi_p(U_p)$ and $\phi_p(W_p)$ are concentric balls centered at the origin. According to the lemma there exist $F_p \in C^\infty(\mathbb{R}^n)$,

$$F_p|_{\phi_p(U_p)} \equiv 1, \quad F_p|_{\mathbb{R}^n - \phi_p(W_p)} \equiv 0.$$

Let

$$f_p(x) = \begin{cases} F_p(\phi_p(x)) & x \in Z_p, \\ 0 & x \notin Z_p. \end{cases}$$

$f_p \in C^\infty(M)$, $f_p|_{U_p} \equiv 1$, $f_p|_{X - W_p} \equiv 0$. Note that A is compact, there exists a finite open cover $\{U_i\}_{1 \le i \le r} \subset \{U_p\}_{p \in A}$. Let

$$f = 1 - (1 - f_1) \cdots (1 - f_r) \in C^\infty(M).$$

If $x \in A$, there must be an i, $x \in U_i$. Then $f_i(x) = 1$, $f(x) \equiv 1$. If $x \notin U$, $f_i(x) = 0$ for $i = 1, \cdots, r$. Then $f(x) \equiv 0$. □

Exercise 21.14. Chern classes of a hypersurface in a complex projective space. Let H be the hyperplane bundle over the projective space $\mathbb{C}P^n$ (see (20.3)), and $H^{\otimes k}$ the tensor product of k copies

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Differential forms in algebraic topology play a crucial role in connecting algebraic structures and geometric intuitions. They provide powerful tools for analyzing properties of manifolds, understanding cohomology theories, and applying integration on differentiable manifolds. This article will explore the fundamentals of differential forms, their applications in algebraic topology, and significant theorems that illustrate their importance.

Understanding Differential Forms

Differential forms are mathematical objects that can be integrated over manifolds. They generalize the notion of functions and can be thought of as a way to encode geometric data. Specifically, differential forms can be defined on smooth manifolds and can be used to study various properties of these spaces.

Definition of Differential Forms

A differential form of degree k on a differentiable manifold M is a smooth section of the bundle of k-forms. Formally, the space of k-forms on M is denoted by $\Omega^k(M)$. A k-form can be expressed locally as:

$$
\omega = f \, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}
$$

where f is a smooth function on M, dx^{i_j} are local coordinate differentials, and $\wedge$ denotes the wedge product, which is an antisymmetric operation.

Key Properties of Differential Forms

1. Linear Structure: The space of k-forms $\Omega^k(M)$ has a vector space structure.
2. Exterior Derivative: For a k-form ω, there exists a differential operator $d: \Omega^k(M) \to \Omega^{k+1}(M)$ called the exterior derivative, which satisfies:
- $d(f) = df$ for $f \in C^\infty(M)$
- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
- $d^2 = 0$
3. Wedge Product: The wedge product of two differential forms is associative and bilinear. The antisymmetry property implies that $\alpha \wedge \alpha = 0$ for any k-form α.

Applications in Algebraic Topology

Differential forms provide important tools for various aspects of algebraic topology, particularly in cohomology theories and integration on manifolds. Below, we explore several significant applications.

De Rham Cohomology

One of the most profound applications of differential forms in algebraic topology is in the formulation of De Rham cohomology. This cohomology theory relates the topology of smooth manifolds to differential forms. The key ideas include:

- Closed and Exact Forms: A differential form ω is called closed if $d\omega = 0$ and exact if $\omega = d\eta$ for some $(k-1)$-form η.
- Cohomology Classes: The k-th De Rham cohomology group $H^k_{dR}(M)$ is defined as the quotient space of closed k-forms modulo exact k-forms:

$$
H^k_{dR}(M) = \frac{\{ \omega \in \Omega^k(M) \mid d\omega = 0 \}}{\{ \omega = d\eta \}}
$$

- Isomorphism Theorem: De Rham's theorem states that the De Rham cohomology groups $H^k_{dR}(M)$ are isomorphic to the singular cohomology groups $H^k(M, \mathbb{R})$. This provides a powerful bridge between differential geometry and algebraic topology.

Integration of Differential Forms

Another critical application of differential forms in algebraic topology is their role in integration on manifolds. The integral of a differential form can be used to compute various topological invariants.

1. Stokes' Theorem: This fundamental theorem relates the integration of differential forms over the boundary of a manifold to the integration over the manifold itself:

$$
\int_{\partial M} \omega = \int_M d\omega
$$

This theorem generalizes the Fundamental Theorem of Calculus and has profound implications in both mathematics and physics.

2. Volume Forms: A top-degree differential form can be used to define a volume form on a manifold. For an n-dimensional manifold M, an n-form ω allows the computation of the volume of M via the integral:

$$
\text{Vol}(M) = \int_M \omega
$$

3. Poincaré Duality: Differential forms are instrumental in establishing Poincaré duality, which states that for a compact oriented manifold M of dimension n, there is an isomorphism:

$$
H^k(M; \mathbb{R}) \cong H_{n-k}(M; \mathbb{R})
$$

This duality reflects a deep relationship between the topology of a manifold and the algebraic structures of its cohomology groups.

Significant Theorems Involving Differential Forms

Several key theorems in algebraic topology highlight the importance of differential forms.

Thom Isomorphism Theorem

The Thom Isomorphism Theorem provides a connection between the topology of a manifold and the cohomology of its submanifolds. It states that the inclusion of a submanifold induces an isomorphism between the cohomology of the manifold and the cohomology of the submanifold, relative to the ambient space.

Whitney's Embedding Theorem

Whitney's Embedding Theorem states that any smooth manifold can be embedded into Euclidean space. This theorem implies that differential forms, which are defined on manifolds, can be studied using the tools of calculus in $\mathbb{R}^n$.

Cartan's Magic Formula

Cartan's Magic Formula relates the exterior derivative and the Lie derivative. For a differential form ω and a vector field X, it is expressed as:

$$
L_X \omega = d(\iota_X \omega) + \iota_X(d\omega)
$$

where L_X denotes the Lie derivative and ι_X represents the interior product. This relationship is essential in differential geometry and theoretical physics, particularly in the context of symplectic geometry and gauge theories.

Conclusion

In conclusion, differential forms in algebraic topology provide a rich framework for understanding the interplay between geometry and topology. Their ability to encapsulate geometric information and facilitate integration makes them invaluable tools in mathematical analysis. From De Rham cohomology to Stokes' Theorem, the applications of differential forms are profound and far-reaching. As we continue to explore the nuances of topology and geometry, differential forms will undoubtedly remain at the forefront of mathematical research and application.

Frequently Asked Questions

What are differential forms and how are they used in algebraic topology?

Differential forms are mathematical objects that generalize the concept of functions and can be integrated over manifolds. In algebraic topology, they are used to define cohomology theories, such as de Rham cohomology, which relates differential forms to topological properties of manifolds.

Can you explain the relationship between differential forms and Stokes' theorem in the context of algebraic topology?

Stokes' theorem is a fundamental result that connects the integration of differential forms over a manifold to the integration over its boundary. In algebraic topology, this theorem is used to define the notion of cohomology and to relate the topology of a manifold to the properties of differential forms defined on it.

What is the significance of de Rham cohomology in algebraic topology?

De Rham cohomology is significant because it provides a way to study the topology of smooth manifolds using differential forms. It allows mathematicians to classify manifolds based on the properties of their differential forms, providing a bridge between differential geometry and algebraic topology.

How do exterior derivatives relate to differential forms and their properties?

The exterior derivative is an operation that extends the concept of differentiation to differential forms. It plays a crucial role in defining the cohomology groups, as it allows the construction of exact sequences that capture topological features of manifolds.

What role do differential forms play in the formulation of the Poincaré duality theorem?

Differential forms are instrumental in the formulation of Poincaré duality, which states that the k-th cohomology group of a closed orientable manifold is isomorphic to the (n-k)-th homology group, where n is the dimension of the manifold. This duality highlights the deep connections between differential forms and the algebraic invariants of topological spaces.

How can one compute the cohomology groups of a manifold using differential forms?

To compute the cohomology groups of a manifold using differential forms, one typically identifies a suitable space of differential forms, applies the exterior derivative to find closed forms, and then uses the concept of exact sequences to relate the closed forms to cohomology classes. This process often involves techniques from both analysis and algebraic topology.

Differential Forms In Algebraic Topology

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The noun form of 'differential' typically refers to differences between amounts of things. For this case, the differential is the different amount between Tom's apples and Jim's apples.

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Đồng nghĩa với different 'Different' may only be an adjective. It describes a lack of similarity. "Tom and Jim are different people." "Tom and Jim each purchased a different number of apples." …

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Explore the role of differential forms in algebraic topology. Uncover their applications and significance in modern mathematics. Learn more in our detailed guide!

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